

# On The Universality Of Central Loops <sup>\*†</sup>

Tèmítópé Gbóláhàn Jaíyéolá  
 Department of Mathematics,  
 Obafemi Awolowo University, Ile Ife, Nigeria.  
 jaiyeolatemitope@yahoo.com, tjayeola@oauife.edu.ng

## Abstract

LC-loops, RC-loops and C-loops are collectively called central loops. It is shown that an LC(RC)-loop is a left(right) universal loop. But an LC(RC)-loop is a universal loop if and only if it is a right(left) universal loop. It is observed that not all RC-loops or LC-loops or C-loops are universal loops. But if an RC-loop(LC-loop, C-loop) is universal, then it is a right Bol loop(left Bol loop, Moufang loop) respectively. If a loop and its right or left isotope are commutative then the loop is a C-loop if and only if its right or left isotope is a C-loop. If a C-loop is central square and its right or left isotope is an alternative central square loop, then the latter is a C-loop. Necessary and sufficient condition for an LC-loop(RC-loop) to be a left(right)G-loop is established. Consequently, necessary and sufficient conditions for an LC-loop, and an RC-loop to be a G-loop are established. A necessary and sufficient condition for a C-loop to be a G-loop is established.

## 1 Introduction

LC-loops, RC-loops and C-loops are loops that respectively satisfy the identities

$$(xx)(yz) = (x(xy))z, \quad (zy)(xx) = z((yx)x) \quad \text{and} \quad x(y(yz)) = ((xy)y)z.$$

These three types of loops will be collectively called central loops. In the theory of loops, central loops are some of the least studied loops. They have been studied by Phillips and Vojtěchovský [23, 21, 22], Kinyon et. al. [24, 17, 18], Ramamurthi and Solarin [25], Fenyves [10] and Beg [4, 5]. The difficulty in studying them is as a result of the nature of the identities defining them when compared with other Bol-Moufang identities. It can be noticed that in the aforementioned LC identity, the two  $x$  variables are consecutively positioned and neither  $y$  nor  $z$  is between them. A similarly observation is true in the other two identities(i.e the

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RC and C identities). But this observation is not true in the identities defining Bol loops, Moufang loops and extra loops. Fenyves [10] gave three equivalent identities that define LC-loops, three equivalent identities that define RC-loops and only one identity that defines C-loops. But recently, Phillips and Vojtěchovský [21, 22] gave four equivalent identities that define LC-loops and four equivalent identities that define RC-loops. Three of the four identities given by Phillips and Vojtěchovský are the same as the three already given by Fenyves and their basic properties are found in [23, 25, 10, 9]. Loops such as Bol loops, Moufang loops, extra loops are the most popular loops of Bol-Moufang type whose isotopic invariance have been considered. But for LC-loops, RC-loops and C-loops, up till this moment, there is no outstanding result on their isotopic invariance.

The left and right translation maps on a loop  $(L, \cdot)$  are the bijections  $L_x : L \rightarrow L$  and  $R_x : L \rightarrow L$ , respectively defined as  $yR_x = y \cdot x = yx$  and  $yL_x = x \cdot y = xy$  for all  $x, y \in L$ . The following subloops of a loop are important for this work.

Let  $(L, \cdot)$  be a loop. The left nucleus of  $L$  is the set

$$N_\lambda(L, \cdot) = \{a \in L : ax \cdot y = a \cdot xy \ \forall x, y \in L\}.$$

The right nucleus of  $L$  is the set

$$N_\rho(L, \cdot) = \{a \in L : y \cdot xa = yx \cdot a \ \forall x, y \in L\}.$$

The middle nucleus of  $L$  is the set

$$N_\mu(L, \cdot) = \{a \in L : ya \cdot x = y \cdot ax \ \forall x, y \in L\}.$$

The nucleus of  $L$  is the set

$$N(L, \cdot) = N_\lambda(L, \cdot) \cap N_\rho(L, \cdot) \cap N_\mu(L, \cdot).$$

The centrum of  $L$  is the set

$$C(L, \cdot) = \{a \in L : ax = xa \ \forall x \in L\}.$$

The center of  $L$  is the set

$$Z(L, \cdot) = N(L, \cdot) \cap C(L, \cdot).$$

$L$  is said to be a centrum square loop if  $x^2 \in C(L, \cdot)$  for all  $x \in L$ .  $L$  is said to be a central square loop if  $x^2 \in Z(L, \cdot)$  for all  $x \in L$ .  $L$  is said to be left alternative if for all  $x, y \in L$ ,  $x \cdot xy = x^2y$  and is said to right alternative if for all  $x, y \in L$ ,  $yx \cdot x = yx^2$ . Thus,  $L$  is said to be alternative if it is both left and right alternative.

The triple  $(U, V, W)$  such that  $U, V, W \in \text{SYM}(L, \cdot)$  is called an autotopism of  $L$  if and only if

$$xU \cdot yV = (x \cdot y)W \ \forall x, y \in L.$$

$\text{SYM}(L, \cdot)$  is called the permutation group of the loop  $(L, \cdot)$ . The group of autotopisms of  $L$  is denoted by  $\text{AUT}(L, \cdot)$ . Let  $(L, \cdot)$  and  $(G, \circ)$  be two distinct loops. The triple

$(U, V, W) : (L, \cdot) \rightarrow (G, \circ)$  such that  $U, V, W : L \rightarrow G$  are bijections is called a loop isotopism if and only if

$$xU \circ yV = (x \cdot y)W \quad \forall x, y \in L.$$

Thus,  $L$  and  $G$  are called loop isotopes. If the triple  $(U, V, I) : \mathcal{G} = (L, \cdot) \rightarrow \mathcal{H} = (L, \circ)$  is an isotopism, then  $\mathcal{H}$  is called a principal isotope of  $\mathcal{G}$ . If  $U = R_g$  and  $V = L_f$ , then  $\mathcal{H}$  is called an  $f, g$ -principal isotope of  $\mathcal{G}$ . The study of  $f, g$ -principal isotopes is important because to every arbitrary isotope  $\mathcal{J} = (G, *)$  of a loop  $\mathcal{G} = (L, \cdot)$ , there exists an  $f, g$ -principal isotope  $\mathcal{H} = (L, \circ)$  of  $\mathcal{G} = (L, \cdot)$  such that  $\mathcal{J} = (G, *) \cong \mathcal{H} = (L, \circ)$ . Hence, to verify the isotopic invariance(or universality) of a loop property, one simply needs to verify the identity in all  $f, g$ -principal isotopes. Therefore, a loop property or identity is isotopic invariant if and only if all its  $f, g$ -principal isotopes obey that property. A loop that is isomorphic to all its loop isotopes is called a G-loop. Thus, a loop is a G-loop if and only if it is isomorphic to all its  $f, g$ -principal isotopes. Some popular G-loops are CC-loops(Goodaire and Robinson [12, 13]), extra loops, K-loops(Basarab [1, 3] but not the K-loops of Kiechle [14]), VD-loops(Basarab [3]) and Buchsteiner loops(Buchsteiner [6], Basarab [2], Csörgő et. al. [8]). Some conditions that characterize a G-loop are highlighted below.

If  $H$  is a loop, then the following conditions are equivalent.

1.  $H$  is a G-loop.
2. Every element  $x \in H$  is a companion of some right and some left pseudo-automorphisms of  $H$ . Pflugfelder [16]
3. There exists a permutation  $\theta$  of  $H$  such that  $(\theta R_x^{-1}, \theta L_y^{-1}, \theta)$  is an autotopism of  $H$  for all  $x, y$  in  $H$ . Chiboka and Solarin [7], Kunen [20]

The study of the  $f, g$ -principal isotopes of central loops is not easy because of the nature of the identities defining them as mentioned earlier. Although its easy to study the  $f, g$ -principal isotopes of some other loops of Bol-Moufang type like Bol loops and Moufang loops. To salvage the situation, the left and right isotopes of central loops will be studied. According to Nagy and Strambach [15], the  $f, e$ -principal isotopes and  $e, g$ -principal isotopes of a loop with identity element  $e$  are respectively called its right and left isotopes. The left and right isotopes of a loop  $(\Omega, \cdot)$  shall be represented by  $(\Omega, e, g)$  and  $(\Omega, f, e)$  respectively.

Most of the expressions that will made in this work are dual in nature i.e if we write 'statement or word or symbol A(B)[C] implies a statement or word or symbol  $\mathfrak{A}(\mathfrak{B})[\mathfrak{C}]$ ' then we mean that 'A implies  $\mathfrak{A}$ ', 'B implies  $\mathfrak{B}$ ' and 'C implies  $\mathfrak{C}$ '.

In this work, it is shown that an LC(RC)-loop is a left(right) universal loop. But an LC(RC)-loop is a universal loop if and only if it is a right(left) universal loop. It is observed that not all RC-loops or LC-loops or C-loops are universal loops. But if an RC-loop(LC-loop, C-loop) is universal, then it is a right Bol loop(left Bol loop, Moufang loop) respectively. If a loop and its right or left isotope are commutative then the loop is a C-loop if and only if its right or left isotope is a C-loop. If a C-loop is central square and its right or left isotope is an alternative central square loop, then the latter is a C-loop. Necessary and sufficient condition

for an LC-loop(RC-loop) to be a left(right)G-loop is established. Consequently, necessary and sufficient conditions for an LC-loop, and an RC-loop to be a G-loop are established. A necessary and sufficient condition for a C-loop to be a G-loop is established.

## 2 Preliminary

**Definition 2.1** *Let the triple  $\alpha = (A, B, C)$  be an isotopism of the groupoid  $(G, \cdot)$  onto a groupoid  $(H, \circ)$ .*

- (a) *If  $\alpha = (A, B, B)$ , then the triple is called a left isotopism and the groupoids are called left isotopes.*
- (b) *If  $\alpha = (A, B, A)$ , then the triple is called a right isotopism and the groupoids are called right isotopes.*
- (c) *If  $\alpha = (A, I, I)$ , then the triple is called a left principal isotopism and the groupoids are called left principal isotopes.*
- (d) *If  $\alpha = (I, B, I)$ , then the triple is called a right principal isotopism and the groupoids are called right principal isotopes.*

**Theorem 2.1** *Let  $(G, \cdot)$  and  $(H, \circ)$  be two distinct left(right) isotopic loops with the former having an identity element  $e$ . For some  $g(f) \in G$ , there exists an  $e, g(f, e)$ -principal isotope  $(G, *)$  of  $(G, \cdot)$  such that  $(H, \circ) \cong (G, *)$ .*

A loop is a left(right) universal "certain" loop if and only if all its left(right) isotopes are "certain" loops. A loop is a universal "certain" loop if and only if it is both a left and a right universal "certain" loop. A loop is called a right G-loop( $G_\rho$ -loop) if and only if it is isomorphic to all its right loop isotopes. A loop is called a left G-loop( $G_\lambda$ -loop) if and only if it is isomorphic to all its left loop isotopes. As shown by Wilson [26], a loop is a G-loop if and only if it is isomorphic to all its right and left isotopes. Thus, a loop is a G-loop if and only if it is a  $G_\rho$ -loop and a  $G_\lambda$ -loop. Kunen [19] demonstrated the use of  $G_\rho$ -loops and  $G_\lambda$ -loops.

**Definition 2.2** *([16], III.3.9 Definition, III.3.10 Definition, III.3.15 Definition)*  
*Let  $(L, \cdot)$  be a loop and  $U, V, W \in \text{SYM}(L, \cdot)$ .*

1. *If  $(U, V, W) \in \text{AUT}(L, \cdot)$  for some  $V, W$ , then  $U$  is called autotopic,*
  - *the set of autotopic bijections in a loop  $(L, \cdot)$  is represented by  $\Sigma(L, \cdot)$ .*
2. *If  $(U, V, W) \in \text{AUT}(L, \cdot)$  such that  $W = U, V = I$ , then  $U$  is called  $\lambda$ -regular,*
  - *the set of all  $\lambda$ -regular bijections in a loop  $(L, \cdot)$  is represented by  $\Lambda(L, \cdot)$ .*
3. *If  $(U, V, W) \in \text{AUT}(L, \cdot)$  such that  $U = I, W = V$ , then  $V$  is called  $\rho$ -regular,*

- the set of all  $\rho$ -regular bijections in a loop  $(L, \cdot)$  is represented by  $P(L, \cdot)$ .
4. If there exists  $V \in \text{SYM}(L, \cdot)$  such that  $xU \cdot y = x \cdot yV$  for all  $x, y \in L$ , then  $U$  is called  $\mu$ -regular while  $U' = V$  is called its adjoint.
- The set of all  $\mu$ -regular bijections in a loop  $(L, \cdot)$  is denoted by  $\Phi(L, \cdot)$ , while the collection of all adjoints in the loop  $(L, \cdot)$  is denoted by  $\Phi^*(L, \cdot)$ .

**Theorem 2.2** ([16], III.3.11 Theorem, III.3.16 Theorem)

The set  $\Lambda(Q, \cdot) \left( P(Q, \cdot) \right) \left[ \Phi(Q, \cdot) \right]$  of all  $\lambda$ -regular ( $\rho$ -regular) [ $\mu$ -regular] bijections of a quasigroup  $(Q, \cdot)$  is a subgroup of the group  $\Sigma(Q, \cdot)$  of all autotopic bijections of  $(Q, \cdot)$ .

**Corollary 2.1** ([16], III.3.12 Corollary, III.3.16 Theorem)

If two quasigroups  $Q$  and  $Q'$  are isotopic, then the corresponding groups  $\Lambda$  and  $\Lambda'$  ( $P$  and  $P'$ ) [ $\Phi$  and  $\Phi'$ ] [ $\Phi^*$  and  $\Phi'^*$ ] are isomorphic.

Throughout this study, the following notations for translations will be adopted;  $L_x : y \mapsto xy$  and  $R_x : y \mapsto yx$  for a loop while  $L'_x : y \mapsto xy$  and  $R'_x : y \mapsto yx$  for its loop isotope.

**Theorem 2.3** A loop  $L$  is an LC-loop if and only if  $(L_x^2, I, L_x^2) \in \text{AUT}(L)$  for all  $x \in L$ .

**Proof**

Let  $L$  be an LC-loop  $\Leftrightarrow (x \cdot xy)z = (xx)(yz) \Leftrightarrow (x \cdot xy)z = x(x \cdot yz)$  by [9]  $\Leftrightarrow (L_x^2, I, L_x^2) \in \text{AUT}(L)$  for all  $x \in L$ .

**Theorem 2.4** A loop  $L$  is an RC-loop if and only if  $(I, R_x^2, R_x^2) \in \text{AUT}(L)$  for all  $x \in L$ .

**Proof**

Let  $L$  be an RC-loop, then  $z(yx \cdot x) = zy \cdot xx \Leftrightarrow y(yx \cdot x) = (zy \cdot x)x$  by [9]  $\Leftrightarrow (I, R_x^2, R_x^2) \in \text{AUT}(L) \forall x \in L$ .

**Lemma 2.1** A loop is an LC(RC)-loop if and only if  $L_x^2(R_x^2)$  is  $\lambda(\rho)$ -regular i.e  $L_x^2(R_x^2) \in \Lambda(L)(P(L))$ .

**Proof**

Using Theorem 2.3(Theorem 2.4), the rest follows from the definition of  $\lambda(\rho)$ -regular bijection.

**Theorem 2.5** A loop  $L$  is a C-loop if and only if  $R_x^2$  is  $\mu$ -regular and the adjoint of  $R_x^2$ , denoted by  $(R_x^2)^* = L_x^2$  i.e  $R_x^2 \in \Phi(L)$  and  $L_x^2 \in \Phi^*(L)$ .

**Proof**

Let  $L$  be a C-loop then  $(yx \cdot x)z = y(x \cdot xz) \Rightarrow yR_x^2 \cdot z = y \cdot zL_x^2 \Rightarrow R_x^2 \in \Phi(L)$  and  $L_x^2 \in \Phi^*(L)$ . Conversely: do the reverse of the above.

**Theorem 2.6** *Let  $G$  be a loop with identity element  $e$  and  $H$  a quasigroup such that they are isotopic under the triple  $\alpha = (A, B, C)$ .*

1. *If  $C = B$ , then  $G \stackrel{A}{\cong} H$  if and only if  $eB \in N_\rho(H)$ .*
2. *If  $C = A$ , then  $G \stackrel{B}{\cong} H$  if and only if  $eA \in N_\lambda(H)$ .*

**Proof**

Here, when  $L_x$  and  $R_x$  are respectively the left and right translations of the loop  $G$  then the left and right translations of its quasigroup isotope  $H$  are denoted by  $L'_x$  and  $R'_x$  respectively.

Let  $(G, \cdot)$  and  $(H, \circ)$  be any two distinct quasigroups. If  $A, B, C : G \rightarrow H$  are permutations, then the following statements are equivalent :

- the triple  $\alpha = (A, B, C)$  is an isotopism of  $G$  upon  $H$ ,

$$R'_{xB} = A^{-1}R_xC \quad \forall x \in G \quad (1)$$

$$L'_{yA} = B^{-1}L_yC \quad \forall y \in G \quad (2)$$

1. When  $\alpha = (A, B, B)$ ,  $R'_{eB} = A^{-1}B \Rightarrow B = AR'_{eB}$ . So,

$$\alpha = (A, AR'_{eB}, AR'_{eB}) = (A, A, A)(I, R'_{eB}, R'_{eB}), \quad \alpha : G \rightarrow H.$$

If  $(A, A, A) : G \rightarrow H$  is an isotopism i.e  $A$  is an isomorphism, then  $(I, R'_{eB}, R'_{eB}) : H \rightarrow H$  is an autotopism if and only if  $eB \in N_\rho(H)$ .

2. When  $\alpha = (A, B, A)$ ,  $L'_{eA} = B^{-1}A \Rightarrow A = BL'_{eA}$ . So,

$$\alpha = (BL'_{eA}, B, BL'_{eA}) = (B, B, B)(L'_{eA}, I, L'_{eA}), \quad \alpha : G \rightarrow H.$$

If  $(B, B, B) : G \rightarrow H$  is an isotopism i.e  $B$  is an isomorphism, then  $(L'_{eA}, I, L'_{eA}) : H \rightarrow H$  is an autotopism if and only if  $eA \in N_\lambda(H)$ .

### 3 Main Results

**Theorem 3.1** *Let  $G = (\Omega, \cdot)$  be a loop with identity element  $e$  such that  $H = (\Omega, \circ) = (\Omega, e, g)$  is any of its left isotopes.  $G$  is an LC-loop if and only if  $H$  is an LC-loop. Furthermore,  $G$  is a  $G_\lambda$ -loop if and only if  $e \in N_\rho(H)$ .*

**Proof**

By Lemma 2.1,  $G$  is an LC-loop  $\Leftrightarrow L_x^2 \in \Lambda(G)$ . From Equation 2 in the proof of Theorem 2.6,  $L'_{xg} = L_x$  for all  $g, x \in G$ . By Corollary 2.1, there exists isomorphisms  $\Lambda(G) \rightarrow \Lambda(H)$ . Thus  $L_y'^2 \in \Lambda(H) \Leftrightarrow H$  is an LC-loop. The converse is proved in a similar way.

To prove the last part, we shall use the first part of Theorem 2.6. Thus,  $G$  is a  $G_\lambda$ -loop if and only if  $e \in N_\rho(H)$ .

**Corollary 3.1** *An LC-loop is a left universal loop.*

**Proof**

This follows from Theorem 3.1.

**Corollary 3.2** *An LC-loop is a*

1. *universal loop if and only if it is a right universal loop.*
2. *G-loop if and only if it is a  $G_\rho$ -loop and its identity element is in the right nucleus of all its left isotopes.*

**Proof**

This follows from Theorem 3.1.

**Theorem 3.2** *Let  $G = (\Omega, \cdot)$  be a loop with identity element  $e$  such that  $H = (\Omega, \circ) = (\Omega, f, e)$  is any of its right isotopes.  $G$  is an RC-loop if and only if  $H$  is an RC-loop. Furthermore,  $G$  is a  $G_\rho$ -loop if and only if  $e \in N_\lambda(H)$ .*

**Proof**

By Lemma 2.1,  $G$  is an RC-loop  $\Leftrightarrow R_x^2 \in P(G)$ . From Equation 1 in the proof of Theorem 2.6,  $R'_{fx} = R_x$  for all  $f, x \in G$ . By Corollary 2.1, there exists isomorphisms  $P(G) \rightarrow P(H)$ . Thus  $R_y'^2 \in P(H) \Leftrightarrow H$  is an RC-loop.

To prove the last part, we shall use the second part of Theorem 2.6. Thus,  $G$  is a  $G_\rho$ -loop if and only if  $e \in N_\lambda(H)$ .

**Corollary 3.3** *An RC-loop is a left universal loop.*

**Proof**

This follows from Theorem 3.2.

**Corollary 3.4** *An RC-loop is a*

1. *universal loop if and only if it is a left universal loop.*
2. *G-loop if and only if it is a  $G_\lambda$ -loop and its identity element is in the left nucleus of all its right isotopes.*

**Proof**

This follows from Theorem 3.2.

**Lemma 3.1** *Not all RC-loops or LC-loops or C-loops are universal loops.*

**Proof**

As shown in Theorem II.3.8 and Theorem II.3.9 of [16], a left(right) inverse property loop is universal if and only if it is a left(right) Bol loop. Not all RC-loops or LC-loops or C-loops are right Bol loops or left Bol loops or Moufang loops respectively. Hence, the proof follows.

**Lemma 3.2** *If an RC-loop(LC-loop, C-loop) is universal, then it is a right Bol loop(left Bol loop, Moufang loop) respectively.*

**Proof**

This follows from the results in Theorem II.3.8 and Theorem II.3.9 of [16] that a left(right) inverse property loop is universal if and only if it is a left(right) Bol loop.

**Remark 3.1** *From Lemma 3.2, it can be inferred that an extra loop is a C-loop and a Moufang loop.*

**Theorem 3.3** *Let  $G = (\Omega, \cdot)$  be a loop with identity element  $e$  such that  $H = (\Omega, \circ) = (\Omega, e, g)$  is any of its left isotopes. If  $G$  is a central square C-loop and  $H$  is an alternative central square loop, then  $H$  is a C-loop. Furthermore,  $G$  is a  $G_\lambda$ -loop if and only if  $e \in N_\rho(H)$ .*

**Proof**

$G$  is a C-loop  $\Leftrightarrow R_x^2 \in \Phi(G)$  and  $(R_x^2)^* = L_x^2 \in \Phi^*(G)$  for all  $x \in G$ . From Equation 2 in the proof of Theorem 2.6,  $L'_{xg} = L_x$  for all  $g, x \in G$ . So using Corollary 2.1,  $R_y'^2 \in \Phi(H)$  and  $(R_y'^2)^* = L_y'^2 \in \Phi^*(H) \Leftrightarrow H$  is a C-loop.

The proof of the last part is similar to the way it was proved in Theorem 3.1.

**Theorem 3.4** *Let  $G = (\Omega, \cdot)$  be a loop with identity element  $e$  such that  $H = (\Omega, \circ) = (\Omega, f, e)$  is any of its right isotopes. If  $G$  is a central square C-loop and  $H$  is an alternative central square loop, then  $H$  is a C-loop. Furthermore,  $G$  is a  $G_\rho$ -loop if and only if  $e \in N_\lambda(H)$ .*

**Proof**

$G$  is a C-loop  $\Leftrightarrow R_x^2 \in \Phi(G)$  and  $(R_x^2)^* = L_x^2 \in \Phi^*(G)$  for all  $x \in G$ . From Equation 1 in the proof of Theorem 2.6,  $R'_{fx} = R_x$  for all  $f, x \in G$ . So using Corollary 2.1,  $R_y'^2 \in \Phi(H)$  and  $(R_y'^2)^* = L_y'^2 \in \Phi^*(H) \Leftrightarrow H$  is a C-loop.

The proof of the last part is similar to the way it was proved in Theorem 3.2.

**Theorem 3.5** *Let  $G = (\Omega, \cdot)$  be a commutative loop with identity element  $e$  such that  $H = (\Omega, \circ) = (\Omega, e, g)$  is any of its commutative left isotopes.  $G$  is a C-loop if and only if  $H$  is a C-loop. Furthermore,  $G$  is a  $G_\lambda$ -loop if and only if  $e \in N_\rho(H)$ .*

**Proof**

The proof is similar to that of Theorem 3.3.

**Theorem 3.6** *Let  $G = (\Omega, \cdot)$  be a commutative loop with identity element  $e$  such that  $H = (\Omega, \circ) = (\Omega, f, e)$  is any of its commutative right isotopes.  $G$  is a C-loop if and only if  $H$  is a C-loop. Furthermore,  $G$  is a  $G_\rho$ -loop if and only if  $e \in N_\lambda(H)$ .*

**Proof**

The proof is similar to that of Theorem 3.3.



**Lemma 3.3** *A central loop is a G-loop if and only if its identity element belongs to the intersection of the left cosets formed by its nucleus.*

**Proof**

Let  $G$  be a central loop. Let its left isotopes be denoted by  $H_i$ ,  $i \in \Pi$  and let its right isotopes be denoted by  $H_j$ ,  $j \in \Delta$ . By Theorem 2.6,  $G$  is a  $G_\lambda$ -loop if and only if  $e \in N_\rho(H_i)$  for all  $i \in \Pi$ . Also,  $G$  is a  $G_\rho$ -loop if and only if  $e \in N_\lambda(H_j)$  for all  $j \in \Delta$ . A loop is a G-loop if and only if it is a  $G_\rho$ -loop and a  $G_\lambda$ -loop. So  $G$  is a G-loop if and only if  $e \in \bigcap_{i \in \Pi, j \in \Delta} (N_\rho(H_i) \cap N_\lambda(H_j))$ .

$N_\rho(H_i)$  and  $N_\lambda(H_j)$  are subgroups for all  $i \in \Pi$  and  $j \in \Delta$ . Let  $G = (\Omega, \cdot)$  so that  $H_i = (\Omega, \circ_i) = (\Omega, e, g_i)$ ,  $i \in \Pi$  and  $H_j = (\Omega, \circ_j) = (\Omega, f_j, e)$ ,  $j \in \Delta$ . Recall that  $N_\rho(G) \xrightarrow{L_{g_i}} N_\rho(H_i)$  for all  $i \in \Pi$  i.e  $N_\rho(G) \cong N_\rho(H_i)$  (III.2.6 Theorem [16]) under the mapping  $L_{g_i}$ . Similarly,  $N_\lambda(G) \xrightarrow{R_{f_j}} N_\lambda(H_j)$  for all  $j \in \Delta$  i.e  $N_\lambda(G) \cong N_\lambda(H_j)$  (III.2.6 Theorem [16]) under the mapping  $R_{f_j}$ . Therefore,  $G$  is a G-loop if and only if  $e \in \bigcap_{i \in \Pi, j \in \Delta} (g_i N_\rho(G) \cap N_\lambda(G) f_j)$ . In the case of  $G$  been a C-loop,  $N(G) = N_\rho(G) = N_\lambda(G)$ ,  $N(G) \triangleleft G$  and  $[G : N(G)] \neq 2$ . So,  $G$  is a G-loop if and only if  $e \in \bigcap_{g \in \{g_i, f_j\}} gN$ .

**Corollary 3.5** *Every alternative central square left(right) isotope  $G$  of a Cayley loop or RA-loop or  $M_{16}(Q_8) \times E \times A$  where  $E$  is an elementary abelian 2-group,  $A$  is an abelian group(all of whose elements have finite odd order) and  $M_{16}(Q_8)$  is a Cayley loop, is a C-loop.*

**Proof**

From [11], the Cayley loop and ring alternative loops(RA-loops) are all central square. Hence, by Theorem 3.3 and Theorem 3.4, the claim that  $G$  is a C-loop follows.

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